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Low-energy dynamics in $N = 2$ super QED: two-loop approximation

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Abstract: The two-loop (Euler-Heisenberg-type) effective action for $\mathcal{N} = 2$ supersymmetric QED is computed using the $\mathcal{N} = 1$ superspace formulation. The effective action is expressed as a series in supersymmetric extensions of F^{2n}, where $n = 2, 3, \ldots$, with F the field strength. The corresponding coefficients are given by triple proper-time integrals which are evaluated exactly. As a by-product, we demonstrate the appearance of a non-vanishing F^4 quantum correction at the two-loop order. The latter result is in conflict with the conclusion of [1] that no such quantum corrections are generated at two loops in generic $\mathcal{N} = 2$ SYM theories on the Coulomb branch. We explain a subtle loophole in the relevant consideration of [1] and re-derive the F^4 term from harmonic supergraphs.

Keywords: Extended Supersymmetry, Supersymmetric Effective Theories, Superspaces.

Contents

1. Introduction and outlook

In our recent paper [2], a manifestly covariant approach was developed for evaluating multi-loop quantum corrections to low-energy effective actions within the background field formulation. This approach is applicable to ordinary gauge theories and to supersymmetric Yang-Mills theories formulated in superspace. Its power is not restricted to computing just the counterterms — it is well suited for deriving finite quantum corrections in the framework of the derivative expansion. More specifically, in the case of supersymmetric Yang-Mills theories, it is free of some drawbacks still present in the classic works [3] (such as the splitting of background covariant derivatives into ordinary derivatives plus the background connection, in the process of evaluating the supergraphs).

As a simple application of the techniques developed in [2], in this note we derive the two-loop (Euler-Heisenberg-type [4]–[6]) effective action for $\mathcal{N} = 2$ supersymmetric QED formulated in $\mathcal{N} = 1$ superspace. This is a supersymmetric generalization of the two-loop QED calculation by Ritus [7] (see also follow-up publications [8]–[12]). It is curious that the two-loop QED effective action [7] was computed only a year after the work by Wess and Zumino [13] that stimulated widespread interest in supersymmetric quantum field theory. To the best of our knowledge, the Ritus results have never been extended before to the supersymmetric case.[1]

Our interest in $\mathcal{N} = 2$ SQED, and not the 'more realistic' $\mathcal{N} = 1$ SQED, is motivated by the fact that there exist numerous (AdS/CFT-correspondence inspired) conjectures about the multi-loop structure of (Coulomb-branch) low-energy actions in *extended* superconformal theories, especially the $\mathcal{N} = 4$ SYM theory, see, e.g. [14] for a discussion and

[1] At the component level, the two-loop effective action (5.22) is not just a combination of the Ritus results for scalars and spinors [7] because of the presence of quartic scalar and Yukawa couplings in $\mathcal{N} = 2$ SQED.

references. None of these conjectures are related to $\mathcal{N}=2$ SQED which is, of course, not a superconformal theory. We believe, nevertheless, that the experience gained and lessons learned through the study of $\mathcal{N}=2$ SQED should be an important stepping stone towards testing these conjectures.

An unexpected outcome of the consideration in this paper concerns one particular conclusion drawn in [1] on the basis of the background field formulation in $\mathcal{N}=2$ harmonic superspace [15]. According to [1], no F^4 quantum correction occurs at two loops in generic $\mathcal{N}=2$ super Yang-Mills theories on the Coulomb branch, in particular in $\mathcal{N}=2$ SQED. However, as it will be shown below, on the basis of the background field formulation in $\mathcal{N}=1$ superspace, there does occur a non-vanishing F^4 two-loop correction in $\mathcal{N}=2$ SQED. Unfortunately, the analysis in [1] turns out to contain a subtle loophole related to the intricate structure of harmonic supergraphs. A more careful treatment of two-loop harmonic supergraphs, which will be given in the present paper, leads to the same non-zero F^4 term in $\mathcal{N}=2$ SQED at two loops as that derived using the $\mathcal{N}=1$ superfield formalism.

Some time ago, Dine and Seiberg [16] argued that the F^4 quantum correction is one-loop exact on the Coulomb branch of $\mathcal{N}=2,4$ superconformal theories. It was also shown [17, 18] that there are no instanton F^4 corrections. The paper [1] provided perturbative two-loop support for the Dine-Seiberg conjecture. Since the two-loop F^4 conclusion of [1] is no longer valid, it would be extremely interesting to carry out an independent calculation of the two-loop F^4 quantum correction in $\mathcal{N}=2$ superconformal theories (it definitely vanishes in $\mathcal{N}=4$ SYM).

The present paper is organized as follows. In section 2 we review, following [2], the structure of exact superpropagators in a covariantly constant $\mathcal{N}=1$ vector multiplet background. Section 3 contains the $\mathcal{N}=2$ SQED setup required for the subsequent consideration. The one-loop effective action for $\mathcal{N}=2$ SQED is reviewed in section 4. The two-loop effective action for $\mathcal{N}=2$ SQED is derived in section 5 — the main original part of this work. In section 6 we re-derive the two-loop F^4 quantum correction using the harmonic superspace formulation for $\mathcal{N}=2$ SQED. The salient properties of the $\mathcal{N}=1$ parallel displacement propagator are collected in appendix.

2. Exact superpropagators

In this section we review, following [2], the structure of exact superpropagators in a covariantly constant $\mathcal{N}=1$ vector multiplet background. Our consideration is not restricted to the U(1) case and is in fact valid for an arbitrary gauge group. The results of this section can be used for loop calculations, in the framework of the background field approach, of special sectors of low-energy effective actions in generic $\mathcal{N}=1$ super Yang-Mills theories. They will be used in the next sections to derive the two-loop (Euler-Heisenberg-type) effective action for $\mathcal{N}=2$ SQED.

Green's functions in $\mathcal{N}=1$ super Yang-Mills theories are typically associated with covariant d'alembertians constructed in terms of the relevant gauge covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha}) = D_A + \mathrm{i}\mathcal{A}_A(z)\,, \tag{2.1}$$

with D_A the flat covariant derivatives,[2] and $\mathcal{A}_A(z)$ the superfield connection taking its values in the Lie algebra of the gauge group. So we start by recalling the algebra of gauge covariant derivatives:

$$
\begin{aligned}
\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0\,, \qquad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -2\mathrm{i}\,\mathcal{D}_{\alpha\dot\beta}\,, \\
[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}] &= 2\mathrm{i}\varepsilon_{\alpha\beta}\,\bar{\mathcal{W}}_{\dot\beta}\,, \qquad [\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\,\mathcal{W}_\beta\,, \\
[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] &= \mathrm{i}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta} = -\varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\,\mathcal{D}_\alpha\mathcal{W}_\beta\,.
\end{aligned} \tag{2.2}
$$

Here the spinor field strengths $\mathcal{W}_\alpha$ and $\bar{\mathcal{W}}_{\dot\alpha}$ obey the Bianchi identities

$$
\bar{\mathcal{D}}_{\dot\alpha}\mathcal{W}_\alpha = 0\,, \qquad \mathcal{D}^\alpha\mathcal{W}_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}\,. \tag{2.3}
$$

There are three major d'alembertians which occur in covariant supergraphs [20]: (i) the vector d'alembertian $\Box_{\rm v}$; (ii) the chiral d'alembertian $\Box_+$; and (iii) the antichiral d'alembertian $\Box_-$. The vector d'alembertian is defined by

$$
\begin{aligned}
\Box_{\rm v} &= \mathcal{D}^a\mathcal{D}_a - \mathcal{W}^\alpha\mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} \\
&= -\frac{1}{8}\mathcal{D}^\alpha\bar{\mathcal{D}}^2\mathcal{D}_\alpha + \frac{1}{16}\{\mathcal{D}^2, \bar{\mathcal{D}}^2\} - \mathcal{W}^\alpha\mathcal{D}_\alpha - \frac{1}{2}(\mathcal{D}^\alpha\mathcal{W}_\alpha) \\
&= -\frac{1}{8}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}^2\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{16}\{\mathcal{D}^2, \bar{\mathcal{D}}^2\} + \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha})\,.
\end{aligned} \tag{2.4}
$$

Among its important properties are the identities

$$
\frac{1}{16}[\mathcal{D}^2, \bar{\mathcal{D}}^2] = \Box_{\rm v} + \frac{\mathrm{i}}{2}\bar{\mathcal{D}}_{\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_\alpha = -\Box_{\rm v} - \frac{\mathrm{i}}{2}\mathcal{D}_\alpha\mathcal{D}^{\alpha\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\,. \tag{2.5}
$$

The covariantly chiral d'alembertian is defined by

$$
\Box_+ = \mathcal{D}^a\mathcal{D}_a - \mathcal{W}^\alpha\mathcal{D}_\alpha - \frac{1}{2}\,(\mathcal{D}^\alpha\mathcal{W}_\alpha)\,, \quad \Box_+\Phi = \frac{1}{16}\,\bar{\mathcal{D}}^2\mathcal{D}^2\Phi\,, \quad \bar{\mathcal{D}}_{\dot\alpha}\Phi = 0\,. \tag{2.6}
$$

As can be seen, the operator $\Box_+$ acts on the space of covariantly chiral superfields. The antichiral d'alembertian is defined similarly,

$$
\Box_- = \mathcal{D}^a\mathcal{D}_a + \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} + \frac{1}{2}\,(\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha})\,, \quad \Box_-\bar\Phi = \frac{1}{16}\,\mathcal{D}^2\bar{\mathcal{D}}^2\bar\Phi\,, \quad \mathcal{D}_\alpha\bar\Phi = 0\,. \tag{2.7}
$$

The operators $\Box_+$ and $\Box_-$ are related to each other as follows:

$$
\mathcal{D}^2\,\Box_+ = \Box_-\,\mathcal{D}^2\,, \qquad \bar{\mathcal{D}}^2\,\Box_- = \Box_+\,\bar{\mathcal{D}}^2\,. \tag{2.8}
$$

Additional relations occur for an on-shell background

$$
\mathcal{D}^\alpha\mathcal{W}_\alpha = 0 \quad \longrightarrow \quad \mathcal{D}^2\,\Box_+ = \mathcal{D}^2\,\Box_{\rm v} = \Box_{\rm v}\,\mathcal{D}^2\,, \quad \bar{\mathcal{D}}^2\,\Box_- = \bar{\mathcal{D}}^2\,\Box_{\rm v} = \Box_{\rm v}\,\bar{\mathcal{D}}^2\,. \tag{2.9}
$$

In what follows, the background vector multiplet is chosen to be covariantly constant and on-shell,

$$
\mathcal{D}_a\mathcal{W}_\beta = 0\,, \qquad \mathcal{D}^\alpha\mathcal{W}_\alpha = 0\,. \tag{2.10}
$$

It is worth noting that the first requirement here implies that the Yang-Mills superfield belongs to the Cartan subalgebra of the gauge group.

[2]Our $\mathcal{N} = 1$ notation and conventions correspond to [19].

Associated with $\Box_{\rm v}$ is a Green's function $G(z,z')$ which is subject to the Feynman boundary conditions and satisfies the equation

$$\left(\Box_{\rm v} - m^2\right) G(z,z') = -\mathbf{1}\,\delta^8(z-z')\,. \qquad (2.11)$$

It possesses the proper-time representation

$$G(z,z') = {\rm i}\int_0^\infty {\rm d}s\, K(z,z'|s)\,{\rm e}^{-{\rm i}(m^2-{\rm i}\varepsilon)s}\,, \qquad \varepsilon \to +0\,. \qquad (2.12)$$

The corresponding heat kernel[3] [2] is

$$K(z,z'|s) = -\frac{\rm i}{(4\pi s)^2}\sqrt{\det\left(\frac{2\,s\,\mathcal{F}}{{\rm e}^{2s\mathcal{F}}-1}\right)}\;{\rm U}(s)\,\zeta^2\bar\zeta^2\,{\rm e}^{\frac{\rm i}{4}\rho\,\mathcal{F}\coth(s\mathcal{F})\,\rho}\,I(z,z')\,, \qquad (2.13)$$

where the determinant is computed with respect to the Lorentz indices,

$${\rm U}(s) = \exp\left\{-{\rm i}s(\mathcal{W}^\alpha\mathcal{D}_\alpha + \bar{\mathcal{W}}^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha})\right\}\,, \qquad (2.14)$$

and $I(z,z')$ is the so-called parallel displacement propagator, see the appendix for its definition and basic properties. The supersymmetric two-point function $\zeta^A(z,z') = -\zeta^A(z',z) = (\rho^a,\zeta^\alpha,\bar\zeta_{\dot\alpha})$ is defined as follows:

$$\rho^a = (x-x')^a - {\rm i}(\theta-\theta')\sigma^a\bar\theta' + {\rm i}\theta'\sigma^a(\bar\theta-\bar\theta')\,, \qquad \zeta^\alpha = (\theta-\theta')^\alpha\,, \qquad \bar\zeta_{\dot\alpha} = (\bar\theta-\bar\theta')_{\dot\alpha}\,. \quad (2.15)$$

Let us introduce proper-time dependent variables $\Psi(s) \equiv {\rm U}(s)\,\Psi\,{\rm U}(-s)$. With the notation

$$\mathcal{N}_\alpha{}^\beta = \mathcal{D}_\alpha\mathcal{W}^\beta\,, \qquad \bar{\mathcal{N}}_{\dot\alpha}{}^{\dot\beta} = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\beta}\,, \qquad {\rm tr}\,\mathcal{N} = {\rm tr}\,\bar{\mathcal{N}} = 0\,, \qquad (2.16)$$

for the buiding blocks appearing in the right hand side of (2.13) we then get

$$\begin{aligned}
\mathcal{W}^\alpha(s) &= (\mathcal{W}\,{\rm e}^{-{\rm i}s\mathcal{N}})^\alpha\,, \qquad \bar{\mathcal{W}}^{\dot\alpha}(s) = (\bar{\mathcal{W}}\,{\rm e}^{-{\rm i}s\bar{\mathcal{N}}})^{\dot\alpha}\,,\\
\zeta^\alpha(s) &= \zeta^\alpha + \left(\mathcal{W}\,\frac{{\rm e}^{-{\rm i}s\mathcal{N}}-1}{\mathcal{N}}\right)^\alpha\,, \qquad \bar\zeta^{\dot\alpha}(s) = \bar\zeta^{\dot\alpha} - \left(\bar{\mathcal{W}}\,\frac{{\rm e}^{-{\rm i}s\bar{\mathcal{N}}}-1}{\bar{\mathcal{N}}}\right)^{\dot\alpha}\,,\\
\rho_{\alpha\dot\alpha}(s) &= \rho_{\alpha\dot\alpha} - 2\int_0^s {\rm d}t\,\Big(\mathcal{W}_\alpha(t)\bar\zeta_{\dot\alpha}(t) + \zeta_\alpha(t)\bar{\mathcal{W}}_{\dot\alpha}(t)\Big)\,. \qquad (2.17)
\end{aligned}$$

One also finds [2]

$$U(s)\,I(z,z') = \exp\left\{\int_0^s {\rm d}t\,\Xi(\zeta(t),\mathcal{W}(t),\bar{\mathcal{W}}(t))\right\} I(z,z')\,, \qquad (2.18)$$

where $\Xi(\zeta(s),\mathcal{W}(s),\bar{\mathcal{W}}(s)) = U(s)\,\Xi(\zeta,\mathcal{W},\bar{\mathcal{W}})\,U(-s)$ and

$$\begin{aligned}
\Xi(\zeta,\mathcal{W},\bar{\mathcal{W}}) &= \frac{1}{12}\,\rho^{\dot\alpha\alpha}\Big(\mathcal{W}^\beta\bar\zeta^{\dot\beta} - \zeta^\beta\bar{\mathcal{W}}^{\dot\beta}\Big)\Big(\varepsilon_{\beta\alpha}\,\bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{W}}_{\dot\alpha} - \varepsilon_{\dot\beta\dot\alpha}\,\mathcal{D}_\beta\mathcal{W}_\alpha\Big) - \frac{2{\rm i}}{3}\,\zeta\mathcal{W}\,\bar\zeta\bar{\mathcal{W}} -\\
&\quad - \frac{\rm i}{3}\,\zeta^2\left(\bar{\mathcal{W}}^2 - \frac14\bar\zeta\bar{\mathcal{D}}\,\bar{\mathcal{W}}^2\right) - \frac{\rm i}{3}\,\bar\zeta^2\left(\mathcal{W}^2 - \frac14\zeta\mathcal{D}\,\mathcal{W}^2\right)\,. \qquad (2.19)
\end{aligned}$$

[3]This heat kernel was first derived in the Fock-Schwinger gauge in [21].

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In the case of a real representation of the gauge group, the Green's function $G(z,z')$ should be realizable as the vacuum average of a time-ordered product,

$$G(z,z') = \mathrm{i}\,\langle 0|T\left(\Sigma(z)\,\Sigma^{\mathrm{T}}(z')\right)|0\rangle \equiv \mathrm{i}\,\langle \Sigma(z)\,\Sigma^{\mathrm{T}}(z')\rangle\,,$$

for a real quantum field $\Sigma(z)$. Therefore the corresponding heat kernel should possess the property

$$K(z',z|s) = K^{\mathrm{T}}(z,z'|s)\,. \tag{2.20}$$

As is seen from (2.13), this property is only obvious for the sub-kernel $\tilde{K}(z,z'|s)$ defined by

$$K(z,z'|s) = \mathrm{U}(s)\,\tilde{K}(z,z'|s)\,. \tag{2.21}$$

However, using the properties of the parallel displacement propagator listed in the appendix, one can show

$$\left(\mathcal{W}^{\alpha}\mathcal{D}_{\alpha} + \bar{\mathcal{W}}^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\right)\tilde{K}(z,z'|s) = \tilde{K}(z,z'|s)\left(\overleftarrow{\mathcal{D}'_{\alpha}}\,\mathcal{W}'^{\alpha} + \overleftarrow{\bar{\mathcal{D}}'_{\dot\alpha}}\,\bar{\mathcal{W}}'^{\dot\alpha}\right), \tag{2.22}$$

and this in fact implies (2.20).

Associated with the chiral d'alembertian $\Box_+$ is a Green's function $G_+(z,z'|s)$ which is covariantly chiral in both arguments,

$$\bar{\mathcal{D}}_{\dot\alpha}\,G_+(z,z') = \bar{\mathcal{D}}'_{\dot\alpha}\,G_+(z,z') = 0\,, \tag{2.23}$$

is subject to the Feynman boundary conditions and satisfies the equation

$$\left(\Box_+ - m^2\right)\,G_+(z,z') = -\mathbf{1}\,\delta_+(z,z')\,, \qquad \mathbf{1}\,\delta_+(z,z') = -\frac{1}{4}\bar{\mathcal{D}}^2\,\mathbf{1}\,\delta^8(z-z')\,. \tag{2.24}$$

Under the restriction $\mathcal{D}^{\alpha}\mathcal{W}_{\alpha} = 0$, this Green's function is related to $G(z,z')$ as follows:

$$G_+(z,z') = -\frac{1}{4}\bar{\mathcal{D}}^2 G(z,z') = -\frac{1}{4}\bar{\mathcal{D}}'^2 G(z,z')\,. \tag{2.25}$$

The corresponding chiral heat kernel[4] turns out to be

$$\begin{aligned} K_+(z,z'|s) \;=\; -\frac{1}{4}\bar{\mathcal{D}}^2 K(z,z'|s) &= -\frac{\mathrm{i}}{(4\pi s)^2}\sqrt{\det\left(\frac{2\,s\,\mathcal{F}}{\mathrm{e}^{2s\mathcal{F}}-1}\right)}\;\mathrm{U}(s)\,\times \\ &\times \zeta^2\,\exp\left(\frac{\mathrm{i}}{4}\rho\,\mathcal{F}\coth(s\mathcal{F})\,\rho - \frac{\mathrm{i}}{2}\rho^a\mathcal{W}\sigma_a\bar{\zeta}\right)\,I(z,z')\,. \end{aligned} \tag{2.26}$$

It is an instructive exercise to check, using the properties of the parallel displacement propagator given in the appendix, that $K_+(z,z'|s)$ is covariantly chiral in both arguments. For completeness, we also present the antichiral-chiral kernel

$$\begin{aligned} \frac{1}{16}\,\mathcal{D}^2\bar{\mathcal{D}}'^2\,K(z,z'|s) \;=\; &-\frac{\mathrm{i}}{(4\pi s)^2}\sqrt{\det\left(\frac{2s\mathcal{F}}{\mathrm{e}^{2s\mathcal{F}}-1}\right)}\;\mathrm{U}(s)\,\times \\ &\times \exp\left(\frac{\mathrm{i}}{4}\tilde{\rho}\,\mathcal{F}\coth(s\mathcal{F})\,\tilde{\rho} + R(z,z')\right)\,I(z,z')\,, \end{aligned} \tag{2.27}$$

[4]In the U(1) case, the chiral heat kernel was first derived in a special gauge in [22].

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where

$$R(z,z') = -\frac{\mathrm{i}}{2}\,\tilde\rho^a(\mathcal{W}\sigma_a\bar\zeta + \zeta\sigma_a\bar{\mathcal{W}}) + \frac{1}{3}(\zeta^2\,\bar\zeta\bar{\mathcal{W}} - \bar\zeta^2\,\zeta\mathcal{W}) + \\ + \frac{\mathrm{i}}{12}\tilde\rho_{\alpha\dot\alpha}(\zeta^\alpha\bar\zeta^{\dot\beta}\,\bar{\mathcal{D}}_{\dot\beta}\bar{\mathcal{W}}^{\dot\alpha} + 5\,\zeta^\beta\bar\zeta^{\dot\alpha}\,\mathcal{D}_\beta\mathcal{W}^\alpha)\,, \tag{2.28}$$

and $\tilde\rho^a$ is a 'left antichiral/right chiral' variable

$$\tilde\rho^a = \rho^a - \mathrm{i}\,\zeta\sigma^a\bar\zeta\,, \qquad \mathcal{D}_\beta\,\tilde\rho^a = \bar{\mathcal{D}}'_{\dot\beta}\,\tilde\rho^a = 0\,. \tag{2.29}$$

The parallel displacement propagator is the only building block for the supersymmetric heat kernels which involves the naked gauge connection. In covariant supergraphs, however, the parallel displacement propagators, that come from all possible internal lines, 'annihilate' each other through the mechanism sketched in [2].

A very special and extremely simple type of background field configuration,

$$\mathcal{D}_\alpha\mathcal{W}_\beta = 0\,, \tag{2.30}$$

is suitable for computing exotic low-energy effective actions of the form

$$\int \mathrm{d}^8z\,L(\mathcal{W},\bar{\mathcal{W}}) + \left(\int \mathrm{d}^6z\,P(\mathcal{W}) + \text{c.c.}\right), \tag{2.31}$$

which are of some interest in the context of the Veneziano-Yankielowicz action [23] and its recent generalizations destined to describe the low-energy dynamics of the glueball superfield $\mathcal{S} = \mathrm{tr}\,\mathcal{W}^2$. Under the constraint (2.30), the kernel (2.13) becomes

$$K(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\mathrm{e}^{\mathrm{i}\rho^2/4s}\,\delta^2(\zeta - \mathrm{i}s\,\mathcal{W})\,\delta^2(\bar\zeta + \mathrm{i}s\,\bar{\mathcal{W}})\,I(z,z')\,, \tag{2.32}$$

while the chiral kernel (2.26) turns into[5]

$$K_+(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\mathrm{e}^{\mathrm{i}\rho^2/4s}\,\delta^2(\zeta - \mathrm{i}s\,\mathcal{W})\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\rho^a\mathcal{W}\sigma_a(\bar\zeta + \mathrm{i}s\,\bar{\mathcal{W}})}\,I(z,z')\,. \tag{2.33}$$

Here the parallel displacement propagator is completely specified by the properties:

$$\begin{aligned} I(z',z)\,\mathcal{D}_{\alpha\dot\alpha}I(z,z') &= -\mathrm{i}(\zeta_\alpha\bar{\mathcal{W}}_{\dot\alpha} + \mathcal{W}_\alpha\,\bar\zeta_{\dot\alpha})\,, \\ I(z',z)\,\mathcal{D}_\alpha I(z,z') &= -\frac{\mathrm{i}}{2}\,\rho_{\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} + \frac{1}{3}(\zeta_\alpha\bar\zeta\bar{\mathcal{W}} + \bar\zeta^2\mathcal{W}_\alpha)\,, \\ I(z',z)\,\bar{\mathcal{D}}_{\dot\alpha}I(z,z') &= -\frac{\mathrm{i}}{2}\,\rho_{\alpha\dot\alpha}\mathcal{W}^\alpha - \frac{1}{3}(\bar\zeta_{\dot\alpha}\zeta\mathcal{W} + \zeta^2\bar{\mathcal{W}}_{\dot\alpha})\,. \end{aligned} \tag{2.34}$$

[5]A simplified version of the chiral kernel (2.33) has recently been used in [24] to provide further support to the Dijkgraaf-Vafa conjecture [25].

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3. $\mathcal{N}=2$ SQED

The action of $\mathcal{N}=2$ SQED written in terms of $\mathcal{N}=1$ superfields is

$$S_{\rm SQED}=\frac{1}{e^2}\int {\rm d}^8z\bar\Phi\Phi+\frac{1}{e^2}\int {\rm d}^6zW^\alpha W_\alpha+\int {\rm d}^8z\Big(\overline{Q}{\rm e}^V Q+\overline{\tilde Q}{\rm e}^{-V}\tilde Q\Big)+\Big({\rm i}\int {\rm d}^6z\tilde Q\Phi Q+{\rm c.c.}\Big)\,, \tag{3.1}$$

where $W_\alpha=-\frac{1}{8}\bar D^2 D_\alpha V$. The dynamical variables Φ and V describe an $\mathcal{N}=2$ abelian vector multiplet, while the superfields Q and $\tilde Q$ constitute a massless Fayet-Sohnius hypermultiplet. The case of a massive hypermultiplet is obtained from (3.1) by the shift $\Phi\to\Phi+m$, with m a complex parameter.[6] Introducing new chiral variables

$$\mathbb{Q}=\exp\Big({\rm i}\frac{\pi}{4}\sigma_1\Big)\begin{pmatrix}Q\\ \tilde Q\end{pmatrix}, \tag{3.2}$$

with $\vec\sigma=(\sigma_1,\sigma_2,\sigma_3)$ the Pauli matrices, the action takes the (real representation) form

$$S_{\rm SQED}=\frac{1}{e^2}\int {\rm d}^8z\bar\Phi\Phi+\frac{1}{e^2}\int {\rm d}^6zW^\alpha W_\alpha+\int {\rm d}^8z\mathbb{Q}^\dagger{\rm e}^{V\sigma_2}\mathbb{Q}+\frac{1}{2}\Big(\int {\rm d}^6z\Phi\mathbb{Q}^{\rm T}\mathbb{Q}+{\rm c.c.}\Big)\,. \tag{3.3}$$

We are interested in a low-energy effective action $\Gamma[W,\Phi]$ which describes the dynamics of the $\mathcal{N}=2$ massless vector multiplet and which is generated by integrating out the massive charged hypermultiplet. More specifically, we concentrate on a slowly varying part of $\Gamma[W,\Phi]$ that, at the component level, comprises contributions with (the supersymmetrization of) all possible powers of the gauge field strength without derivatives. Its generic form is [29]

$$\Gamma[W,\Phi]=\Big(\alpha\int {\rm d}^6z\,W^2\,\ln\frac{\Phi}{\mu}+{\rm c.c.}\Big)+\int {\rm d}^8z\,\frac{\bar W^2W^2}{\bar\Phi^2\Phi^2}\,\Omega(\Psi^2,\bar\Psi^2)\,, \tag{3.4}$$

where

$$\bar\Psi^2=\frac{1}{4}D^2\Big(\frac{W^2}{\bar\Phi^2\Phi^2}\Big)\,,\qquad \Psi^2=\frac{1}{4}\bar D^2\Big(\frac{\bar W^2}{\bar\Phi^2\Phi^2}\Big)\,, \tag{3.5}$$

μ is the renormalization scale and Ω some real analytic function. The first term on the right hand side of (3.4) is known to be one-loop exact in perturbation theory, while the second term receives quantum corrections at all loops.

To evaluate quantum loop corrections to the effective action (3.4), we use the $\mathcal{N}=1$ superfield background field method in its simplest realization, as we are dealing with an abelian gauge theory. Let us split the dynamical variables as follows: $\Phi\to\Phi+\varphi$, $V\to V+v$, $\mathbb{Q}\to\mathbb{Q}+\mathbf{q}$, where Φ, V and $\mathbb{Q}$ are background superfields, while φ,v and $\mathbf{q}$ are quantum ones. As is standard in the background field approach, (background covariant) gauge conditions are to be introduced for the quantum gauge freedom while keeping intact the background gauge invariance. Since we are only interested in the quantum corrections of the form (3.4), it is sufficient to consider simple background configurations

$$\partial_a W_\beta=D^\alpha W_\alpha=0\,,\qquad D_\alpha\Phi=0\,,\qquad \mathbb{Q}=0\,. \tag{3.6}$$

[6] The action of $\mathcal{N}=1$ SQED is obtained from (3.1) by discarding Φ as a dynamical variable, and instead 'freezing' Φ to a constant value m.

Upon quantization in Feynman gauge, we end up with the following action to be used for loop calculations

$$
\begin{aligned}
S_{\text{quantum}} &= \frac{1}{e^2}\int \mathrm{d}^8 z\, \left(\bar{\varphi}\varphi - \frac{1}{2}v\Box v\right) + \\
&+ \int \mathrm{d}^8 z\, \mathbf{q}^\dagger \mathrm{e}^{v\,\sigma_2}\mathbf{q} + \frac{1}{2}\left(\int \mathrm{d}^6 z\,(\Phi+\varphi)\,\mathbf{q}^{\mathrm{T}}\mathbf{q} + \text{c.c.}\right)\,, \qquad (3.7)
\end{aligned}
$$

with $\Box = \partial^a\partial_a$. It is understood here that the quantum superfields $\mathbf{q}$ and $\mathbf{q}^\dagger$ are background covariantly chiral and antichiral, respectively,

$$
\bar{\mathcal{D}}_{\dot\alpha}\mathbf{q} = 0\,, \qquad \mathcal{D}_\alpha \mathbf{q}^\dagger = 0\,. \qquad (3.8)
$$

From the quadratic part of (3.7) one reads off the Feynman propagators

$$
\begin{aligned}
\mathrm{i}\,\langle \mathbf{q}(z)\mathbf{q}^\dagger(z')\rangle &= \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}'^2\, G(z,z')\,,\\
\mathrm{i}\,\langle \mathbf{q}(z)\mathbf{q}^{\mathrm{T}}(z')\rangle &= \frac{1}{4}\bar\Phi\,\bar{\mathcal{D}}^2 G(z,z') = \frac{1}{4}\bar\Phi\,\bar{\mathcal{D}}'^2 G(z,z')\,,\\
\mathrm{i}\,\langle \varphi(z)\bar\varphi(z')\rangle &= -\frac{e^2}{16}\bar{D}^2 D'^2\,\frac{1}{\Box}\,\delta^8(z-z') = -\frac{\mathrm{i}}{16}\bar{D}^2 D'^2\,\langle v(z)v(z')\rangle\,,\\
\mathrm{i}\,\langle v(z)v(z')\rangle &= \frac{e^2}{\Box}\,\delta^8(z-z') = -\frac{e^2}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}u}{u^2}\,\mathrm{e}^{\mathrm{i}\rho^2/4u}\,\delta^2(\zeta)\,\delta^2(\bar\zeta)\,. \qquad (3.9)
\end{aligned}
$$

Here the Green's function $G(z,z')$ transforms in the defining representation of $\mathrm{SO}(2)\cong \mathrm{U}(1)$, and satisfies the equation (2.11) with $m^2 = \bar\Phi\Phi$. It is given by the proper-time representation (2.12) with the heat kernel $K(z,z'|s)$ specified in (2.13). It is understood that the field strengths $\mathcal{W}_\alpha$, $\bar{\mathcal{W}}_{\dot\alpha}$ and their covariant derivatives (such as $\mathcal{F}_{ab}$) are related to W_α, $\bar{W}_{\dot\alpha}$ as follows

$$
\mathcal{W}_\alpha = W_\alpha\,\sigma_2\,, \qquad \bar{\mathcal{W}}_{\dot\alpha} = \bar{W}_{\dot\alpha}\,\sigma_2\,. \qquad (3.10)
$$

4. One-loop effective action

For the sake of completeness, we discuss here the structure of the one-loop effective action [21] and [26]–[29]. Its formal representation is (see [19] for more details)

$$
\Gamma_{\text{one-loop}} = -\frac{\mathrm{i}}{2}\mu^{2\omega}\int_0^\infty \frac{\mathrm{d}(\mathrm{i}s)}{(\mathrm{i}s)^{1-\omega}}\;\mathrm{Tr}\;K_+(s)\,\mathrm{e}^{-\mathrm{i}(\bar\Phi\Phi-\mathrm{i}\varepsilon)s}\,, \qquad (4.1)
$$

where ω is the regularization parameter ($\omega\to 0$ at the end of calculation), and μ the normalization point. The functional trace of the chiral kernel is defined by

$$
\mathrm{Tr}\,K_+(s) = \int \mathrm{d}^6 z\;\mathrm{tr}\;K_+(z,z|s)\,. \qquad (4.2)
$$

Using the explicit form of the chiral kernel (2.26), we obtain

$$
\frac{1}{2}\,\mathrm{tr}\,K_+(z,z|s) = \frac{\mathrm{i}}{(4\pi)^2}\,W^2\,\frac{\sin^2(sB/2)}{(sB/2)^2}\,\sqrt{\det\left(\frac{sF}{\sinh(sF)}\right)}\,, \qquad (4.3)
$$

where we have introduced the notation

$$B^2 = \frac{1}{2}\,\mathrm{tr}\,N^2\,, \quad N_\alpha{}^\beta = D_\alpha W^\beta\,; \qquad \bar{B}^2 = \frac{1}{2}\,\mathrm{tr}\,\bar{N}^2\,, \qquad \bar{N}_{\dot\alpha}{}^{\dot\beta} = \bar{D}_{\dot\alpha}\bar{W}^{\dot\beta}\,. \tag{4.4}$$

For the background superfields under consideration, we have

$$B^2 = \frac{1}{4}D^2W^2\,, \qquad \bar{B}^2 = \frac{1}{4}\bar{D}^2\bar{W}^2\,. \tag{4.5}$$

The latter objects turn out to appear as building blocks for the eigenvalues of $F = (F_a{}^b)$ which are equal to $\pm\lambda_+$ and $\pm\lambda_-$, where

$$\lambda_\pm = \frac{\mathrm{i}}{2}(B \pm \bar{B})\,, \qquad 2B^2 = F^{ab}F_{ab} + \frac{\mathrm{i}}{2}\varepsilon^{abcd}F_{ab}F_{cd}\,. \tag{4.6}$$

This gives

$$\sqrt{\det\left(\frac{sF}{\sinh(sF)}\right)} = \frac{s\lambda_+}{\sinh(s\lambda_+)}\,\frac{s\lambda_-}{\sinh(s\lambda_-)}\,. \tag{4.7}$$

Now, the effective action takes the form

$$\Gamma_{\mathrm{one-loop}} = \frac{\mu^{2\omega}}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d(i}s)}{(\mathrm{i}s)^{1-\omega}}\int \mathrm{d}^6z\, W^2\,\Upsilon(sB, s\bar{B})\,\mathrm{e}^{-\mathrm{i}(\bar\Phi\Phi-\mathrm{i}\varepsilon)s}\,, \tag{4.8}$$

where

$$\Upsilon(x,y) = \frac{\cos x - 1}{x^2}\,\frac{x^2-y^2}{\cos x - \cos y}\,, \qquad \Upsilon(x,0) = 0\,. \tag{4.9}$$

Introducing a new function $\zeta(x,y)$ related to Υ by [29]

$$\Upsilon(x,y) - 1 = -y^2\,\zeta(x,y)\,, \quad \zeta(x,y) = \zeta(y,x) = \frac{y^2(\cos x - 1) - x^2(\cos y - 1)}{x^2y^2(\cos x - \cos y)} \tag{4.10}$$

allows one to readily separate a UV divergent contribution and to represent the finite part of the effective action as an integral over the full superspace. Making use of eq. (4.5) and the standard identity

$$-\frac{1}{4}\int \mathrm{d}^6z\,\bar{D}^2\,L = \int \mathrm{d}^8z\,L\,,$$

for the renormalized one-loop effective action[7] one ends up with

$$\begin{aligned}\Gamma_{\mathrm{one-loop}} = &-\frac{1}{(4\pi)^2}\int \mathrm{d}^6z\,W^2\,\ln\frac{\Phi}{\mu} \;+\; \mathrm{c.c.} + \\ &+\frac{1}{(4\pi)^2}\int \mathrm{d}^8z\,\frac{\bar{W}^2W^2}{\bar\Phi^2\Phi^2}\int_0^\infty \mathrm{d}s\,s\,\zeta(s\Psi, s\bar\Psi)\,\mathrm{e}^{-\mathrm{i}(1-\mathrm{i}\varepsilon)s}\,,\end{aligned} \tag{4.11}$$

with Ψ and $\bar\Psi$ defined in (3.5).

[7]In deriving the effective action (4.11), we concentrated on the quantum corrections involving the $\mathcal{N} = 1$ vector multiplet field strength and did not take into account the effective Kähler potential $K(\bar\Phi,\Phi) = -\frac{1}{(4\pi)^2}\bar\Phi\Phi\ln(\bar\Phi\Phi/\mu^2) = \bar\Phi\mathcal{F}'(\Phi) + \Phi\bar{\mathcal{F}}'(\bar\Phi)$ generated by the holomorphic Seiberg potential $\mathcal{F}(\Phi) = -\frac{1}{(4\pi)^2}\Phi^2\ln(\Phi/\mu)$, as well as higher derivative quantum corrections with chiral superfields. A derivation of $K(\bar\Phi,\Phi)$ using the superfield proper-time technique was first given in [30, 19], see also more recent calculations [31, 32] based on conventional supergraph techniques. The leading higher derivative quantum correction with chiral superfields was computed in [30].

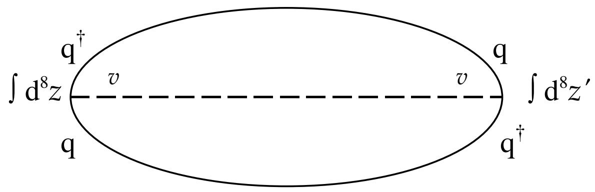

Figure 1: Two-loop supergraph I

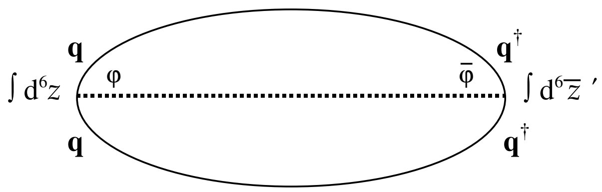

Figure 2: Two-loop supergraph II.

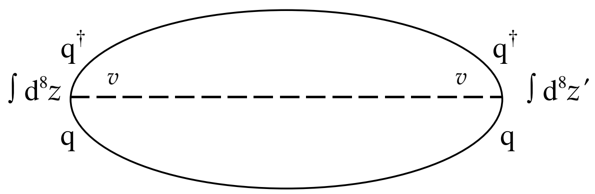

Figure 3: Two-loop supergraph III.

5. Two-loop effective action

We now turn to computing the two-loop quantum correction to the effective action. There are three supergraphs contributing at two loops,[8], and they are depicted in figures 1–3.

The contribution from the first two supergraphs is

$$\Gamma_{\mathrm{I+II}} = \frac{\mathrm{i}}{2^9} \int \mathrm{d}^8 z \int \mathrm{d}^8 z' \, \langle v(z) v(z') \rangle \, \mathrm{tr} \left\{ \left(\bar{\mathcal{D}}^2 \mathcal{D}^2 \, G(z,z') \right) \, [\bar{\mathcal{D}}^2, \mathcal{D}^2] \, G(z',z) \right\} . \qquad (5.1)$$

The third supergraph leads to the following contribution

$$\Gamma_{\mathrm{III}} = \frac{\mathrm{i}}{2^5} \int \mathrm{d}^8 z \int \mathrm{d}^8 z' \, \langle v(z) v(z') \rangle \, \bar{\Phi} \Phi \; \mathrm{tr} \left\{ \left(\bar{\mathcal{D}}^2 \, G(z,z') \right) \, \mathcal{D}^2 \, G(z',z) \right\} . \qquad (5.2)$$

It turns out that the expression for $\Gamma_{\mathrm{I+II}}$ can be considerably simplified using the properties of the superpropagators and their heat kernels, which were discussed in section 2. Since

$$\bar{\mathcal{D}}^2 \, G(z,z') = \bar{\mathcal{D}}'^2 \, G(z,z') \, , \qquad \mathcal{D}^2 \, G(z,z') = \mathcal{D}'^2 \, G(z,z') \, ,$$

we have

$$\bar{\mathcal{D}}^2 \mathcal{D}^2 \, G(z,z') = \mathcal{D}'^2 \bar{\mathcal{D}}'^2 \, G(z,z') \, ,$$

[8] There is actually one more two-loop supergraph, the so-called ‘eight’ diagram, generated by the quartic vertex $\frac{1}{2} \int \mathrm{d}^8 z \, \mathbf{q}^\dagger v^2 \mathbf{q}$; its contribution is obviously zero.

and therefore

$$[\bar{\mathcal{D}}^2, \mathcal{D}^2]\, G(z,z') = -[\bar{\mathcal{D}}'^2, \mathcal{D}'^2]\, G(z,z')\,. \tag{5.3}$$

The latter relation in conjunction with the symmetry property

$$\langle v(z) v(z')\rangle = \langle v(z') v(z)\rangle\,, \qquad G(z,z') = \Big(G(z',z)\Big)^{\mathrm{T}} \tag{5.4}$$

leads to the new representation for $\Gamma_{\mathrm{I+II}}$

$$\Gamma_{\mathrm{I+II}} = \frac{\mathrm{i}}{2^{10}} \int \mathrm{d}^8 z \int \mathrm{d}^8 z'\, \langle v(z) v(z')\rangle\ \mathrm{tr}\, \Big\{ \Big([\bar{\mathcal{D}}^2, \mathcal{D}^2]\, G(z,z')\Big)\ [\bar{\mathcal{D}}^2, \mathcal{D}^2]\, G(z',z)\Big\}\,. \tag{5.5}$$

In accordance with (2.5), we can represent

$$\frac{1}{16}[\mathcal{D}^2, \bar{\mathcal{D}}^2] = \frac{\mathrm{i}}{4} \bar{\mathcal{D}}_{\dot\alpha} \mathcal{D}^{\alpha\dot\alpha} \mathcal{D}_\alpha - \frac{\mathrm{i}}{4} \mathcal{D}_\alpha \mathcal{D}^{\alpha\dot\alpha} \bar{\mathcal{D}}_{\dot\alpha}\,, \tag{5.6}$$

and this identity turns out to be very useful when computing the action of the commutators of covariant derivatives in (5.5) on the Green's functions. A direct evaluation gives

$$\begin{aligned} \frac{1}{16}[\bar{\mathcal{D}}^2, \mathcal{D}^2]\, K(z,z'|s) \approx \frac{\mathrm{i}}{(4\pi s)^2} \sqrt{\det\left(\frac{2\, s\, \mathcal{F}}{\mathrm{e}^{2s\mathcal{F}}-1}\right)} \left(\rho\, \frac{2\mathcal{F}}{\mathrm{e}^{2s\mathcal{F}}-1}\right)^{\alpha\dot\alpha} \zeta_\alpha(s) \bar\zeta_{\dot\alpha}(s) \times \\ \times\, \mathrm{e}^{\frac{\mathrm{i}}{4}\rho\, \mathcal{F} \coth(s\mathcal{F})\, \rho}\, I(z,z')\,, \end{aligned} \tag{5.7}$$

where we have omitted all terms of at least third order in the Grassmann variables ζ_α, $\bar\zeta_{\dot\alpha}$ and $\mathcal{W}_\alpha$, $\bar{\mathcal{W}}_{\dot\alpha}$ as they do not contribute to (5.5). It is easy to derive

$$\left(\rho\, \frac{2\mathcal{F}}{\mathrm{e}^{2s\mathcal{F}}-1}\right)^{\alpha\dot\alpha} \zeta_\alpha(s) \bar\zeta_{\dot\alpha}(s) \Big|_{\zeta=\bar\zeta=0} = \rho\, \frac{F}{\sinh(sF)}\, \frac{\sinh(sF_+)}{F_+}\, \frac{\sinh(sF_-)}{F_-} \Psi\, \mathbf{1}_2\,, \tag{5.8}$$

where

$$\Psi^a = W \sigma^a \bar W\,, \qquad F_\pm = \frac{1}{2}(F \pm \mathrm{i}\, \tilde F)\,, \tag{5.9}$$

with $\tilde F$ the Hodge-dual of F. Here we have taken into account the fact that $\mathcal{F} = F\, \sigma_2$.

As the propagator $\langle v(z) v(z')\rangle$ contains the Grassmann delta-function $\delta^2(\zeta)\delta^2(\bar\zeta)$, the integral over θ' in (5.5) can be trivially done. Replacing the bosonic integration variables in (5.5) by the rule $\{x,\, x'\} \to \{x,\, \rho\}$, as inspired by [7], we end up with

$$\begin{aligned} \Gamma_{\mathrm{I+II}} = \frac{4e^2}{(4\pi)^6} \int \mathrm{d}^8 z\, W^2 \bar W^2 \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t \int_0^\infty \frac{\mathrm{d}u}{u^2} \sqrt{\det\left(\frac{sF}{\sinh(sF)}\frac{tF}{\sinh(tF)}\right)} \times \\ \times \frac{\sin(sB/2)}{sB}\, \frac{\sin(s\bar B/2)}{s\bar B}\, \frac{\sin(tB/2)}{tB}\, \frac{\sin(t\bar B/2)}{t\bar B}\, \mathrm{e}^{-\mathrm{i}(\bar\Phi\Phi - \mathrm{i}\varepsilon)(s+t)} \\ \times \int \mathrm{d}^4\rho\, \left(\rho\, \frac{F}{\sinh(sF)}\, \frac{F}{\sinh(tF)}\, \rho\right) \mathrm{e}^{\mathrm{i}\rho A \rho/4}\,, \end{aligned} \tag{5.10}$$

where

$$A = F\coth(sF) + F\coth(tF) + \frac{1}{u}\,. \tag{5.11}$$

The parallel displacement propagators that come from the two Green's functions in (5.5) annihilate each other, in accordance with (A.5).

Using the explicit structure of the chiral kernel (2.26), it is easy to calculate the contribution from the third supergraph

$$
\begin{aligned}
\Gamma_{\rm III} = & -\frac{8e^2}{(4\pi)^6}\int \mathrm{d}^8z\, W^2\bar{W}^2 \int\limits_0^\infty \mathrm{d}s \int\limits_0^\infty \mathrm{d}t \int\limits_0^\infty \frac{\mathrm{d}u}{u^2}\sqrt{\det\left(\frac{sF}{\sinh(sF)}\frac{tF}{\sinh(tF)}\right)}\times \\
& \times\left\{\frac{\sin^2(sB/2)}{(sB)^2}\,\frac{\sin^2(t\bar{B}/2)}{(t\bar{B})^2} + (s\leftrightarrow t)\right\}\bar{\Phi}\Phi\, \mathrm{e}^{-\mathrm{i}(\bar{\Phi}\Phi - \mathrm{i}\varepsilon)(s+t)}\times \\
& \times\int \mathrm{d}^4\rho\, \mathrm{e}^{\mathrm{i}\rho A\rho/4}\,.
\end{aligned}
\tag{5.12}
$$

Following the non-supersymmetric consideration of Ritus [7], it is useful to introduce the generating functional of gaussian moments

$$
Z(p) = \frac{1}{(4\pi)^2}\int \mathrm{d}^4\rho \exp\left\{\frac{\mathrm{i}}{4}\rho_a\, A^a{}_b\, \rho^b + \mathrm{i}p_a\,\rho^a\right\} = \frac{\mathrm{i}}{\sqrt{\det A}}\,\mathrm{e}^{-\mathrm{i}p\,A^{-1}\,p}\,, \tag{5.13}
$$

where A is defined in (5.11) and is such that $\eta A = (\eta_{ab}\, A^b{}_c)$ is symmetric, with η_{ab} the Minkowski metric. From this we get two important special cases:

$$
\begin{aligned}
\frac{1}{(4\pi)^2}\int \mathrm{d}^4\rho\, \mathrm{e}^{\mathrm{i}\rho A\rho/4} &= Z(0) = \frac{\mathrm{i}}{\sqrt{\det A}}\,, \\
\frac{1}{(4\pi)^2}\int \mathrm{d}^4\rho\, \rho_a\rho_b\, \mathrm{e}^{\mathrm{i}\rho A\rho/4} &= -\frac{\partial^2}{\partial p^a\partial p^b}\, Z(p)\Big|_{p=0} = -\frac{2}{\sqrt{\det A}}\,(A^{-1})_{ab}\,.
\end{aligned}
\tag{5.14}
$$

These results allow us to do the gaussian ρ-integrals in (5.10) and (5.12).

As a next step, we have to compute the determinant of A, with A defined in (5.11), as well as the expression

$$
\mathrm{tr}\left[\frac{F}{\sinh(sF)}\,\frac{F}{\sinh(tF)}\,A^{-1}\right]
$$

which appears in (5.10) after having done the ρ-integral. Recalling the eigenvalues of $F = (F_a{}^b)$ given in eq. (4.6), we obtain

$$
\frac{1}{\sqrt{\det A}} = \frac{1}{(u^{-1}+a_+)(u^{-1}+a_-)}\,, \tag{5.15}
$$

where

$$
a_\pm = \lambda_\pm \coth(s\lambda_\pm) + \lambda_\pm \coth(t\lambda_\pm)\,. \tag{5.16}
$$

With the notation

$$
P_\pm = \frac{1}{st}\,\frac{s\lambda_\pm}{\sinh(s\lambda_\pm)}\,\frac{t\lambda_\pm}{\sinh(t\lambda_\pm)}\,, \tag{5.17}
$$

we also get

$$
\frac{1}{2}\,\mathrm{tr}\left[\frac{F}{\sinh(sF)}\,\frac{F}{\sinh(tF)}\,\frac{1}{F\coth(sF)+F\coth(tF)+u^{-1}}\right] = \frac{P_+}{u^{-1}+a_+} + \frac{P_-}{u^{-1}+a_-}\,. \tag{5.18}
$$

With the old result (4.7), all the building blocks in (5.10) and (5.12) thus become simple functions of the B and $\bar{B}$.

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The proper-time u-integrals in (5.10) and (5.12) are identical to the ones considererd by Ritus [7]. Two integrals occur

$$I_1(s,t) = \int_0^\infty \frac{\mathrm{d}u}{u^2} \frac{1}{(u^{-1}+a_+)(u^{-1}+a_-)}\,, \tag{5.19}$$

$$I_2(s,t) = \int_0^\infty \frac{\mathrm{d}u}{u^2} \frac{1}{(u^{-1}+a_+)(u^{-1}+a_-)} \left(\frac{P_+}{u^{-1}+a_+} + \frac{P_-}{u^{-1}+a_-}\right), \tag{5.20}$$

and their direct evaluation gives

$$\begin{aligned} I_1(s,t) &= \frac{1}{a_+ - a_-} \ln\left(\frac{a_+}{a_-}\right), \\ I_2(s,t) &= \frac{1}{a_+ - a_-} \left(\frac{P_-}{a_-} - \frac{P_+}{a_+}\right) + \frac{P_+ - P_-}{(a_+ - a_-)^2} \ln\left(\frac{a_+}{a_-}\right). \end{aligned} \tag{5.21}$$

However, the expressions obtained do not make manifest the fact that the two-loop effective action

$$\begin{aligned} \Gamma_{\text{two-loop}} = \Gamma_{\mathrm{I+II}} + \Gamma_{\mathrm{III}} = -\frac{e^2}{(4\pi)^4} \int \mathrm{d}^8 z\, W^2 \bar{W}^2 \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}(\bar\Phi\Phi - \mathrm{i}\varepsilon)(s+t)} \times \\ \times \frac{s\lambda_+}{\sinh(s\lambda_+)} \frac{s\lambda_-}{\sinh(s\lambda_-)} \frac{t\lambda_+}{\sinh(t\lambda_+)} \frac{t\lambda_-}{\sinh(t\lambda_-)} \times \\ \times \left(\frac{\sin(sB/2)}{sB/2} \frac{\sin(s\bar B/2)}{s\bar B/2} \frac{\sin(tB/2)}{tB/2} \frac{\sin(t\bar B/2)}{t\bar B/2} I_2(s,t) + \right. \\ \left. + \frac{\mathrm{i}}{2} \bar\Phi\Phi \left\{ \frac{\sin^2(sB/2)}{(sB/2)^2} \frac{\sin^2(t\bar B/2)}{(t\bar B/2)^2} + (s \leftrightarrow t) \right\} I_1(s,t) \right) \end{aligned} \tag{5.22}$$

is free of any divergences, unlike the two-loop QED effective action [7]. This is why we would like to describe a different approach to computing the proper-time integrals, which is most efficient for evaluating effective actions in the framework of the derivative expansion.

The integrands in (5.19) and (5.20) involve two or three factors of $(u^{-1}+a_\pm)^{-1}$, with $a_\pm$ defined in (5.16). With the notation $x = st/u$, one can represent

$$\frac{1}{(u^{-1}+a_\pm)} = \frac{st}{x+s+t} \left\{ 1 + \sum_{n=1}^\infty (-1)^n \left(\frac{st}{x+s+t}\right)^n \Big(\mathcal{L}_\pm(s) + \mathcal{L}_\pm(t)\Big)^n \right\}, \tag{5.23}$$

where

$$\mathcal{L}_\pm(s) = \lambda_\pm \coth(s\lambda_\pm) - \frac{1}{s} \tag{5.24}$$

is regular at $s = 0$. Using these decompositions and replacing the integration variable $u \to x = st/u$, one can easily do the integrals (5.19) and (5.20). Now, if one takes into account the explicit form of $P_\pm$, see eq. (5.17), as well as the structure of the effective action (5.22), it is easy to see that all the remaining proper-time s- and t-integrals are of the following generic form (after the Wick rotation $s = -\mathrm{i}\tilde s$ and $t = -\mathrm{i}\tilde t$)

$$\int_0^\infty \mathrm{d}\tilde s \int_0^\infty \mathrm{d}\tilde t\, \frac{\tilde s^m \tilde t^n}{(\tilde s + \tilde t)^p} \mathrm{e}^{-\mu(\tilde s + \tilde t)} = \frac{(m+n+1-p)!\, m!\, n!}{(m+n+1)!} \frac{1}{\mu^{m+n+2-p}}\,, \qquad \mu > 0\,, \tag{5.25}$$

with m, n and p non-negative integers such that $p \le m+n+1$.

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Recently, Dunne and Schubert [33] obtained closed-form expressions for the two-loop scalar and spinor QED effective lagrangians in the case of a slowly varying *self-dual* background. In the supersymmetric case, the effective action vanishes for a self-dual vector multiplet. Nevertheless, the results of [33] may be helpful in order to obtain a closed-form expression for a holomorphic part of the two-loop effective action (5.22)

$$\Gamma_{\text{holomorphic}} = \int \mathrm{d}^8 z \, \frac{\bar{W}^2 W^2}{\bar{\Phi}^2 \Phi^2} \, \{\Lambda(\Psi^2) \ + \ \bar{\Lambda}(\bar{\Psi}^2)\} \,, \tag{5.26}$$

with Ψ^2 and $\bar{\Psi}^2$ defined in (3.5).

The effective action (5.22) contains supersymmetric extensions of the terms F^{2n}, where $n = 2, 3, \ldots$, with F the electromagnetic field strength. Of special importance is the leading F^4 quantum correction, whose manifestly supersymmetric form is

$$c \int \mathrm{d}^8 z \, \frac{\bar{W}^2 W^2}{\bar{\Phi}^2 \Phi^2} \,. \tag{5.27}$$

It can be singled out from (5.22) by considering the limit B, $\bar{B} \to 0$ in conjunction with

$$I_1(s,t) \to \int_0^\infty \frac{\mathrm{d}u}{u^2} \frac{1}{(u^{-1} + s^{-1} + t^{-1})^2} \,, \qquad I_2(s,t) \to \frac{2}{st} \int_0^\infty \frac{\mathrm{d}u}{u^2} \frac{1}{(u^{-1} + s^{-1} + t^{-1})^3} \,.$$

Direct evaluation, with use of (5.25), gives

$$c_{\text{two-loop}} = \frac{e^2}{2(4\pi)^4} \,. \tag{5.28}$$

This result turns out to be in conflict with a prediction made in [1] on the basis of the background field formulation in $\mathcal{N} = 2$ harmonic superspace [15]. According to [1], no F^4 quantum correction occurs at two loops in generic $\mathcal{N} = 2$ super Yang-Mills theories on the Coulomb branch.

Unfortunately, the consideration of [1] contains a subtle loophole. Its origin will be uncovered in the next section. It will also be shown that a careful evaluation of two-loop $\mathcal{N} = 2$ harmonic supergraphs leads to the same result (5.28) we have just obtained from $\mathcal{N} = 1$ superfields.

6. The two-loop F^4 quantum correction from harmonic supergraphs

In this section, we will re-derive the two-loop F^4 quantum correction using an off-shell formulation for $\mathcal{N} = 2$ SQED in harmonic superspace [35].

The $\mathcal{N} = 2$ harmonic superspace $\mathbb{R}^{4|8} \times S^2$ extends conventional superspace, with coordinates $z^M = (x^m, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}})$, where $i = \underline{1}, \underline{2}$, by the two-sphere $S^2 = \mathrm{SU}(2)/\,\mathrm{U}(1)$ parametrized by harmonics, i.e., group elements

$$(u_i{}^-\,,\, u_i{}^+) \in \mathrm{SU}(2)\,, \qquad u^+_i = \varepsilon_{ij} u^{+j}\,, \qquad \overline{u^{+i}} = u^-_i\,, \quad u^{+i} u^-_i = 1\,. \tag{6.1}$$

The main conceptual advantage of harmonic superspace is that both the $\mathcal{N} = 2$ Yang-Mills vector multiplets and hypermultiplets can be described by *unconstrained* superfields over the analytic subspace of $\mathbb{R}^{4|8} \times S^2$ parametrized by the variables $\zeta^{\mathcal{M}} \equiv (x^m_A, \theta^{+\alpha}, \bar{\theta}^+_{\dot{\alpha}},$

u_i^+, u_j^-), where the so-called analytic basis is defined by

$$x_A^m = x^m + 2\mathrm{i}\,\theta^{(i}\sigma^m\bar\theta^{j)}u_i^+u_j^- \,, \qquad \theta_\alpha^\pm = u_i^\pm\theta_\alpha^i \,, \qquad \bar\theta_{\dot\alpha}^\pm = u_i^\pm\bar\theta_{\dot\alpha}^i \,. \tag{6.2}$$

The $\mathcal{N}=2$ abelian vector multiplet is described by a real analytic superfield $V^{++}(\zeta)$. The charged hypermultiplet is described by an analytic superfield $Q^+(\zeta)$ and its conjugate $\breve{Q}^+(\zeta)$. The classical action for $\mathcal{N}=2$ SQED is

$$S_{\mathrm{SQED}} = \frac{1}{2e^2}\int \mathrm{d}^4x\mathrm{d}^4\theta\, W^2 - \int \mathrm{d}\zeta^{(-4)}\,\breve{Q}^+\mathcal{D}^{++}Q^+ \,. \tag{6.3}$$

Here $W(z)$ is the $\mathcal{N}=2$ chiral superfield strength [36], $\mathrm{d}\zeta^{(-4)}$ denotes the analytic subspace integration measure, and the harmonic (analyticity-preserving) covariant derivative is $\mathcal{D}^{++} = D^{++} \pm \mathrm{i}\,V^{++}$ when acting on Q^+ and $\breve{Q}^+$, respectively. The vector multiplet kinetic term in (6.3) can be expressed as a gauge invariant functional of V^{++} [37].

Upon quantization in the background field approach [15], the quantum theory is governed by the action (lower-case letters are used for the quantum superfields)

$$S_{\mathrm{quantum}} = \frac{1}{2e^2}\int \mathrm{d}\zeta^{(-4)}\, v^{++}\Box\, v^{++} - \int \mathrm{d}\zeta^{(-4)}\,\breve{q}^+\left(\mathcal{D}^{++} + \mathrm{i}\,v^{++}\right)q^+ \,, \tag{6.4}$$

which has to be used for loop calculations. The relevant Feynman propagators [35, 15] are

$$\begin{aligned}
\mathrm{i}\,\langle q^+(\zeta_1)\,\breve{q}^+(\zeta_2)\rangle &= \frac{1}{\widehat{\Box}_1}\,(\mathcal{D}_1^+)^4\,(\mathcal{D}_2^+)^4\,\delta^{12}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^3}\,,\\
\mathrm{i}\,\langle v^{++}(\zeta_1)\,v^{++}(\zeta_2)\rangle &= -\frac{e^2}{\Box_1}\,\delta_{\mathrm{A}}^{(2,2)}(\zeta_1,\zeta_2)\\
&= \frac{e^2}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,(D_1^+)^4\,\mathrm{e}^{\mathrm{i}\rho^2/4s}\,\delta^8(\theta_1-\theta_2)\,\delta^{(-2,2)}(u_1,u_2)
\end{aligned} \tag{6.5}$$

with $\delta_{\mathrm{A}}^{(2,2)}(\zeta_1,\zeta_2)$ the analytic delta-function [35],

$$\delta_{\mathrm{A}}^{(2,2)}(\zeta_1,\zeta_2) = (D_1^+)^4\,\delta^{12}(z_1-z_2)\,\delta^{(-2,2)}(u_1,u_2)\,. \tag{6.6}$$

Here the two-point function ρ^a is defined similarly to its $\mathcal{N}=1$ counterpart (2.15). The covariantly analytic d'alembertian [15] is

$$\begin{aligned}
\widehat{\Box} = \mathcal{D}^m\mathcal{D}_m &- \frac{\mathrm{i}}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^- - \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}_{\dot\alpha}^+\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha} + \frac{\mathrm{i}}{4}(\mathcal{D}^{+\alpha}\mathcal{D}_\alpha^+\mathcal{W})\mathcal{D}^{--} -\\
&- \frac{\mathrm{i}}{8}[\mathcal{D}^{+\alpha},\mathcal{D}_\alpha^-]\mathcal{W} - \bar{\mathcal{W}}\mathcal{W}\,,
\end{aligned} \tag{6.7}$$

where $\mathcal{W} = \pm W$ when acting on q^+ and $\breve{q}^+$, respectively. The algebra of $\mathcal{N}=2$ gauge covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \mathcal{D}_j^{\dot\alpha}) = D_A + \mathrm{i}\,\mathcal{A}_A$ derived in [36] can be expressed in the form

$$\begin{aligned}
\{\mathcal{D}_\alpha^+,\mathcal{D}_\beta^+\} &= \{\bar{\mathcal{D}}_{\dot\alpha}^+,\bar{\mathcal{D}}_{\dot\beta}^+\} = \{\mathcal{D}_\alpha^+,\bar{\mathcal{D}}_{\dot\beta}^+\} = 0\,,\\
\{\bar{\mathcal{D}}_{\dot\alpha}^+,\mathcal{D}_\alpha^-\} &= -\{\mathcal{D}_\alpha^+,\bar{\mathcal{D}}_{\dot\alpha}^-\} = 2\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\,,\\
\{\mathcal{D}_\alpha^+,\mathcal{D}_\beta^-\} &= -2\mathrm{i}\,\varepsilon_{\alpha\beta}\bar{\mathcal{W}}\,, \qquad \{\bar{\mathcal{D}}_{\dot\alpha}^+,\bar{\mathcal{D}}_{\dot\beta}^-\} = 2\mathrm{i}\,\varepsilon_{\dot\alpha\dot\beta}\mathcal{W}\,,
\end{aligned} \tag{6.8}$$

where $\mathcal{D}_\alpha^\pm = \mathcal{D}_\alpha^i\,u_i^\pm$ and $\bar{\mathcal{D}}_{\dot\alpha}^\pm = \bar{\mathcal{D}}_{\dot\alpha}^i\,u_i^\pm$.

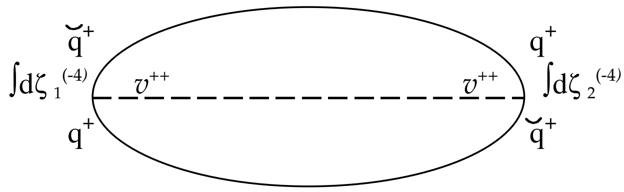

Figure 4: Two-loop harmonic supergraph.

Let us recall the argument given in [1] that *no* non-holomorphic quantum corrections of the form

$$\int \mathrm{d}^{12}z\, H(W,\bar{W}) \equiv \int \mathrm{d}^4x \mathrm{d}^8\theta\, H(W,\bar{W}) \tag{6.9}$$

occur at two loops. By definition, the two-loop effective action is

$$\Gamma_{\text{two-loop}} = \frac{\mathrm{i}^3}{2}\int \mathrm{d}\zeta_1^{(-4)} \int \mathrm{d}\zeta_2^{(-4)}\, \langle v^{++}(1)\, v^{++}(2)\rangle\, \langle q^+(1)\, \breve{q}^+(2)\rangle\, \langle q^+(2)\, \breve{q}^+(1)\rangle\,, \tag{6.10}$$

and it is generated by a single supergraph depicted in figure 4.

Following [1], the crucial step is to lift the analytic subspace integrals to those over the full superspace, by representing, say, $\langle q^+(1)\, \breve{q}^+(2)\rangle$ in the form

$$\langle q^+(1)\, \breve{q}^+(2)\rangle = (\mathcal{D}_1^+)^4\, (\mathcal{D}_2^+)^4\, A^{(-3,-3)}(1,2)\,, \qquad A^{(-3,-3)}(1,2) = \frac{1}{\widehat{\Box}_1}\, \frac{\delta^{12}(z_1 - z_2)}{(u_1^+ u_2^+)^3}\,, \tag{6.11}$$

and then using the standard identity

$$\int \mathrm{d}\zeta^{(-4)}\, (D^+)^4\, L(z,u) = \int \mathrm{d}^{12}z \mathrm{d}u\, L(z,u)\,. \tag{6.12}$$

Since we are only after the quantum correction (6.9), it now suffices to approximate, in the resulting two-loop expression

$$\int \mathrm{d}^{12}z_1 \mathrm{d}u_1 \int \mathrm{d}^{12}z_2 \mathrm{d}u_2\, \langle v^{++}(1)\, v^{++}(2)\rangle\, A^{(-3,-3)}(1,2) \langle q^+(2)\, \breve{q}^+(1)\rangle\,, \tag{6.13}$$

the covariantly analytic d'alembertian by a free massive one,

$$\widehat{\Box} \approx \Box - \bar{W}W\,. \tag{6.14}$$

Now, the part of the integrand in (6.13), which involves the Grassmann delta-functions and spinor covariant derivatives, becomes

$$\delta^8(\theta_1 - \theta_2)\, \left\{(D_1^+)^4 \delta^8(\theta_1 - \theta_2)\right\}\, \left\{(D_1^+)^4 (D_2^+)^4 \delta^8(\theta_1 - \theta_2)\right\}, \tag{6.15}$$

and this expression is obviously zero. Therefore, one naturally concludes $H(W,\bar{W}) = 0$.

Unfortunately, there is a subtle loophole in the above consideration. The point is that upon removing the two factors of $(D^+)^4$ from the hypermultiplet propagator (6.11), in order to convert analytic integrals into full superspace integrals, we apparently end up with a more singular harmonic distribution, $A^{(-3,-3)}(1,2)$, than the original propagator. As a

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result, the expression (6.13) contains the product of two harmonic distributions

$$\delta^{(-2,2)}(u_1,u_2)\,\frac{1}{(u_1^+u_2^+)^3}\,, \tag{6.16}$$

and such a product is ill-defined. To make the consideration sensible, we have to regularize the harmonic distributions $\delta^{(-2,2)}(u_1,u_2)$ and $(u_1^+u_2^+)^{-3}$ from the very beginning. However, the analytic delta-function (6.6) is known to be analytic in both arguments only if the right hand side involves the *genuine* harmonic delta-function, see [35] for more details. With a regularized harmonic delta-function, however, one has to use a modified (but equivalent) expression for the analytic delta-function [35]

$$\begin{aligned}
\delta_{\mathrm{A}}^{(2,2)}(\zeta_1,\zeta_2) &= \frac{1}{2\Box_1}(D_1^+)^4\,(D_2^+)^4\,(D_2^{--})^2\,\delta^{12}(z_1-z_2)\,\delta^{(-2,2)}(u_1,u_2)\\
&= \frac{1}{2\Box_1}(D_1^+)^4\,(D_2^+)^4\,(D_1^{--})^2\,\delta^{12}(z_1-z_2)\,\delta^{(2,-2)}(u_1,u_2)\,.
\end{aligned} \tag{6.17}$$

This expression is good in the sense that it allows for a regularized nonsingular harmonic delta-function. But it is more singular in space-time than (6.6) — an additional source for infrared problems in quantum theory, as will be demonstrated shortly.

Using the alternative representation (6.17) for the analytic delta-function, we would like to undertake a second attempt to evaluate $H(W,\bar W)$. Let us start again with the expression (6.10) for $\Gamma_{\text{two-loop}}$ in which the gluon propagator now reads

$$\mathrm{i}\,\langle v^{++}(1)\,v^{++}(2)\rangle = -\frac{e^2}{2(\Box_1)^2}\,(D_1^+)^4\,(D_2^+)^4\,(D_1^{--})^2\,\delta^{12}(z_1-z_2)\,\delta^{(2,-2)}(u_1,u_2)\,. \tag{6.18}$$

In contrast to the previous consideration, we now make use of the two factors of $(D^+)^4$ from $\langle v^{++}(1)\,v^{++}(2)\rangle$ in order to convert the analytic subspace integrals into ones over the full superspace, thus leaving the hypermultiplet propagators intact. Such a procedure will lead, up to an overall numerical factor, to

$$\begin{aligned}
&\int \mathrm{d}^{12}z_1\mathrm{d}u_1 \int \mathrm{d}^{12}z_2\mathrm{d}u_2\,\left\{\frac{1}{(\Box_1)^2}\,\delta^{12}(z_1-z_2)\right\}\times\\
&\times\delta^{(2,-2)}(u_1,u_2)\,(D_1^{--})^2\,\left\{\langle q^+(1)\,\breve{q}^+(2)\rangle\,\langle q^+(2)\,\breve{q}^+(1)\rangle\right\}.
\end{aligned} \tag{6.19}$$

This does not seem to be identically zero and, in fact, can easily be evaluated. The crucial step is to make use of the identity [34]

$$\begin{aligned}
&(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\,\frac{\delta^{12}(z_1-z_2)}{(u_1^+u_2^+)^3} = \\
&= (\mathcal{D}_1^+)^4\,\left\{(\mathcal{D}_1^-)^4\,(u_1^+u_2^+) - \frac{\mathrm{i}}{2}\,\Delta_1^{--}\,(u_1^-u_2^+) - \widehat{\Box}_1\,\frac{(u_1^-u_2^+)^2}{(u_1^+u_2^+)}\right\}\,\delta^{12}(z_1-z_2)\,,
\end{aligned} \tag{6.20}$$

where

$$\begin{aligned}
\Delta^{--} &= \mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_\alpha^-\bar{\mathcal{D}}_{\dot\alpha}^- + \frac{1}{2}\mathcal{W}(\mathcal{D}^-)^2 + \frac{1}{2}\bar{\mathcal{W}}(\bar{\mathcal{D}}^-)^2 +\\
&+(\mathcal{D}^-\mathcal{W})\mathcal{D}^- + (\bar{\mathcal{D}}^-\bar{\mathcal{W}})\bar{\mathcal{D}}^- + \frac{1}{2}(\mathcal{D}^-\mathcal{D}^-\mathcal{W})\,.
\end{aligned} \tag{6.21}$$

If we are only after $H(W,\bar{W})$, the covariantly analytic d'alembertian can again be approximated as in (6.14). Because of the Grassmann delta-function in the first line of (6.19), only the first term in the right-hand side of (6.20) may produce a non-vanishing contribution. With the harmonic identities

$$(u_1^+ u_2^+)|_{1=2} = 0\,, \qquad D_1^{--}(u_1^+ u_2^+) = (u_1^- u_2^+)\,, \qquad (u_1^- u_2^+)|_{1=2} = -1\,, \tag{6.22}$$

it can be seen that $H(W,\bar{W})$ is determined by the momentum integral

$$H(W,\bar{W}) \propto \int \mathrm{d}^4p \int \mathrm{d}^4k \, \frac{1}{(p^2+\bar{W}W)\,(k^2+\bar{W}W)\,(p+k)^4}\,. \tag{6.23}$$

The bad news is that this integral is both UV and IR divergent. This is the price one has to pay for having made use of the IR-unsafe representation (6.17).

It is of course possible to regularize the integral (6.23) and, then, extract a finite part. Instead of practising black magic, however, we would like to present one more calculation that will lead to a manifestly finite and well-defined expression for $H(W,\bar{W})$. The idea is to take seriously the representation (6.10) and stay in the analytic subspace at all stages of the calculation, without artificial conversion of analytic integrals into those over the full superspace (and without use of the IR-unsafe representation (6.17)). Instead of computing the contribution (6.9) directly, in such a setup we should actually look for an equivalent higher-derivative quantum correction of the form

$$\int \mathrm{d}\zeta^{(-4)}\,(D^+)^4 H(W,\bar{W})\,. \tag{6.24}$$

We are going to work with an on-shell $\mathcal{N}=2$ vector multiplet background

$$D^+D^+W = 0\,. \tag{6.25}$$

In the analytic basis, the delta-function (6.6) can be represented as [35]

$$\delta_{\mathrm{A}}^{(2,2)}(\zeta_1,\zeta_2) = \delta^4(x_1-x_2)\left(\theta_1^+ - (u_1^+u_2^-)\theta_2^+\right)^4 \delta^{(-2,2)}(u_1,u_2)\,. \tag{6.26}$$

Let us use this expression for $\delta_{\mathrm{A}}^{(2,2)}(\zeta_1,\zeta_2)$ in the gluon propagator $\langle v^{++}(\zeta_1)\,v^{++}(\zeta_2)\rangle$, as defined in eq. (6.5), which appears in the effective action (6.10). It is obvious that the operator $(1/\Box_1)$ acts on $\delta^4(x_1-x_2)$ only. The Grassmann delta-function, $(\theta_1^+ - (u_1^+u_2^-)\theta_2^+)^4$, can be used to do one of the Grassmann integrals in (6.10). Similarly, the harmonic delta-function, $\delta^{(-2,2)}(u_1,u_2)$, can be used to do one of the harmonic integrals in (6.10). As a result, the hypermultiplet propagators in (6.10) should be evaluated in the following coincidence limit: $\theta_1=\theta_2$ and $u_1=u_2$. To implement this limit, it is again advantageous to make use of the identity (6.20). It is not difficult to see that only the second term on the right of (6.20) can contribute. Each term in the operator Δ^{--}, (6.21), contains two spinor derivatives. Taken together with the overall factor $(\mathcal{D}^+)^4$ in (6.20), we have a total of six spinor derivatives. But we need eight such derivatives to annihilate the spinor delta-function $\delta^8(\theta_1-\theta_2)$ entering each hypermultiplet propagator. Two missing derivatives come from the covariantly analytic d'alembertian. Introducing the Fock-Schwinger proper-time representation

$$-\frac{1}{\widehat{\Box}} = \mathrm{i} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\, s\, \widehat{\Box}}\,, \tag{6.27}$$

it turns out to be sufficient to approximate

$$\mathrm{e}^{\mathrm{i}\, s \widehat{\Box}} \approx \frac{1}{2}\left(\frac{s}{2}\right)^2 \left\{(D^{+\alpha}W)\mathcal{D}^-_\alpha + (\bar{D}^+_{\dot\alpha}\bar{W})\bar{\mathcal{D}}^{-\dot\alpha}\right\}^2 \mathrm{e}^{\mathrm{i}\, s(\Box - \bar{W}W)}\,. \tag{6.28}$$

After that, it only remains to apply the identity

$$(D^+)^4\,(D^-)^4\,\delta^8(\theta-\theta')\Big|_{\theta=\theta'} = 1 \tag{6.29}$$

in order to complete the D-algebra gymnastics. The remaining technical steps (i.e. the calculation of gaussian space-time integrals and of triple proper-time integrals) are identical to those described before in the $\mathcal{N}=1$ case. Therefore, we simply give the final result for the quantum correction under consideration:

$$\begin{aligned}\frac{e^2}{32(4\pi)^4}\int \mathrm{d}\zeta^{(-4)}\,\frac{(D^+W)^2(\bar{D}^+\bar{W})^2}{(W\bar{W})^2} &= \frac{e^2}{2(4\pi)^4}\int \mathrm{d}\zeta^{(-4)}\,(D^+)^4\Big(\ln W\,\ln\bar{W}\Big) \\ &= \frac{e^2}{2(4\pi)^4}\int \mathrm{d}^4x\mathrm{d}^8\theta\,\ln W\,\ln\bar{W}\,. \end{aligned}\tag{6.30}$$

Upon reduction to $\mathcal{N}=1$ superspace, this functional can be shown to take the form (5.27) with the coefficient c equal to (5.28). The reduction to $\mathcal{N}=1$ superfields is defined as usual: $U| = (x,\theta_i,\bar\theta^j)|_{\theta_2=\bar\theta^2=0}$, for any $\mathcal{N}=2$ superfield U. The $\mathcal{N}=1$ components of W are

$$\Phi = W|\,, \qquad -2\mathrm{i}\,W_\alpha = D^2_\alpha\,W|\,. \tag{6.31}$$

Harmonic superspace still remains to be tamed for quantum practitioners, and the present situation is reminiscent of that with QED in the mid 1940's. It is worth hoping that, as with QED, it should take no longer than half a decade of development for this approach to become a safe and indispensable scheme for quantum calculations in $\mathcal{N}=2$ SYM theories.

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A. Parallel displacement propagator

In this appendix we describe, following [2], the salient properties of the $\mathcal{N}=1$ parallel displacement propagator $I(z,z')$. This object is uniquely specified by the following requirements:

(i) the gauge transformation law

$$I(z,z') \rightarrow e^{i\tau(z)} I(z,z') e^{-i\tau(z')} \tag{A.1}$$

with respect to an arbitrary gauge (τ-frame) transformation of the covariant derivatives

$$\mathcal{D}_A \rightarrow e^{i\tau(z)} \mathcal{D}_A e^{-i\tau(z)} , \qquad \tau^\dagger = \tau , \tag{A.2}$$

with the gauge parameter $\tau(z)$ being arbitrary modulo the reality condition imposed;

(ii) the equation

$$\zeta^A \mathcal{D}_A I(z,z') = \zeta^A \Big(D_A + i\,\mathcal{A}_A(z)\Big) I(z,z') = 0 ; \tag{A.3}$$

(iii) the boundary condition

$$I(z,z) = \mathbf{1} . \tag{A.4}$$

These imply the important relation

$$I(z,z')\, I(z',z) = \mathbf{1} , \tag{A.5}$$

as well as

$$\zeta^A \mathcal{D}'_A I(z,z') = \zeta^A \left(D'_A I(z,z') - i\, I(z,z')\, \mathcal{A}_A(z')\right) = 0 . \tag{A.6}$$

Under hermitean conjugation, the parallel displacement propagator transforms as

$$\big(I(z,z')\big)^\dagger = I(z',z) . \tag{A.7}$$

For a covariantly constant vector multiplet,

$$\mathcal{D}_a W_\beta = 0 , \tag{A.8}$$

the covariant differentiation of $\mathcal{D}_A I(z,z')$ gives [2]

$$\begin{aligned}
\mathcal{D}_{\beta\dot\beta} I(z,z') &= I(z,z') \Big(-\frac{i}{4} \rho^{\dot\alpha\alpha} \mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z') - i\,\zeta_\beta \bar{\mathcal{W}}_{\dot\beta}(z') + i\,\bar\zeta_{\dot\beta} \mathcal{W}_\beta(z') + \\
&\qquad + \frac{2i}{3} \bar\zeta_{\dot\beta} \zeta^\alpha \mathcal{D}_\alpha \mathcal{W}_\beta(z') + \frac{2i}{3} \zeta_\beta \bar\zeta^{\dot\alpha} \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{W}}_{\dot\beta}(z') \Big) \\
&= \Big(-\frac{i}{4} \rho^{\dot\alpha\alpha} \mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z) - i\,\zeta_\beta \bar{\mathcal{W}}_{\dot\beta}(z) + i\,\bar\zeta_{\dot\beta} \mathcal{W}_\beta(z) - \\
&\qquad - \frac{i}{3} \bar\zeta_{\dot\beta} \zeta^\alpha \mathcal{D}_\alpha \mathcal{W}_\beta(z) - \frac{i}{3} \zeta_\beta \bar\zeta^{\dot\alpha} \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{W}}_{\dot\beta}(z) \Big) I(z,z') ;
\end{aligned} \tag{A.9}$$

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$$\begin{aligned}
\mathcal{D}_\beta I(z,z') &= I(z,z')\left(\frac{1}{12}\,\bar\zeta^{\dot\beta}\rho^{\alpha\dot\alpha}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z') - \mathrm{i}\,\rho_{\beta\dot\beta}\Big\{\frac{1}{2}\bar{\mathcal{W}}^{\dot\beta}(z') - \frac{1}{3}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\beta}(z')\Big\} + \right.\\
&\qquad \left. + \frac{1}{3}\zeta_\beta\bar\zeta_{\dot\beta}\bar{W}^{\dot\beta}(z') + \frac{1}{3}\bar\zeta^2\Big\{\mathcal{W}_\beta(z') + \frac{1}{2}\zeta^\alpha\mathcal{D}_\alpha\mathcal{W}_\beta(z') - \frac{1}{4}\zeta_\beta\mathcal{D}^\alpha\mathcal{W}_\alpha(z')\Big\}\right)\\
&= \left(\frac{1}{12}\,\bar\zeta^{\dot\beta}\rho^{\alpha\dot\alpha}\,\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z) - \frac{\mathrm{i}}{2}\,\rho_{\beta\dot\beta}\Big\{\bar{\mathcal{W}}^{\dot\beta}(z) + \frac{1}{3}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\beta}(z)\Big\} + \frac{1}{3}\,\zeta_\beta\bar\zeta_{\dot\beta}\bar{\mathcal{W}}^{\dot\beta}(z) + \right.\\
&\qquad \left. + \frac{1}{3}\bar\zeta^2\Big\{\mathcal{W}_\beta(z) - \frac{1}{2}\zeta^\alpha\mathcal{D}_\alpha\mathcal{W}_\beta(z) + \frac{1}{4}\zeta_\beta\mathcal{D}^\alpha\mathcal{W}_\alpha(z)\Big\}\right) I(z,z')\,; \qquad \text{(A.10)}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{D}}_{\dot\beta} I(z,z') &= I(z,z')\left(-\frac{1}{12}\,\zeta^{\beta}\rho^{\alpha\dot\alpha}\,F_{\alpha\dot\alpha,\beta\dot\beta}(z') - \mathrm{i}\,\rho_{\beta\dot\beta}\Big\{\frac{1}{2}\mathcal{W}^{\beta}(z') + \frac{1}{3}\zeta^{\alpha}\mathcal{D}_{\alpha}\mathcal{W}^{\beta}(z')\Big\} - \right.\\
&\qquad \left. - \frac{1}{3}\bar\zeta_{\dot\beta}\zeta^\beta\mathcal{W}_\beta(z') - \frac{1}{3}\zeta^2\Big\{\bar{\mathcal{W}}_{\dot\beta}(z') - \frac{1}{2}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta}(z') + \frac{1}{4}\bar\zeta_{\dot\beta}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha}(z')\Big\}\right)\\
&= \left(-\frac{1}{12}\,\zeta^{\beta}\rho^{\alpha\dot\alpha}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z) - \frac{\mathrm{i}}{2}\,\rho_{\beta\dot\beta}\Big\{\mathcal{W}^{\beta}(z) - \frac{1}{3}\zeta^{\alpha}\mathcal{D}_{\alpha}\mathcal{W}^{\beta}(z)\Big\} - \frac{1}{3}\,\bar\zeta_{\dot\beta}\zeta^\beta\mathcal{W}_\beta(z) - \right.\\
&\qquad \left. - \frac{1}{3}\zeta^2\Big\{\bar{\mathcal{W}}_{\dot\beta}(z) + \frac{1}{2}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta}(z) - \frac{1}{4}\bar\zeta_{\dot\beta}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha}(z)\Big\}\right) I(z,z')\,. \qquad \text{(A.11)}
\end{aligned}$$

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